Combination of GPS and Leveling Observations and Geoid Models Using Least-Squares Variance Component Estimation

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Abstract: In the last few years, satellite-based positioning techniques, such as the Global Positioning System (GPS), proved their abilities in various geodetic fields. Height determination, as one of such applications, requires a more likely correct geoid model to provide reliable geoid heights for transformation of the ellipsoidal heights to orthometric heights. An important step is then to establish such a model by optimal combination of the available geoid models. This can be achieved via variance component estimation (VCE) methods, which provide appropriate weights to GPS and leveling observations as well as the geoid models. The authors demonstrate the efficacy of the least-squares VCE (LS-VCE) to this problem. The algorithm is applied to real data sets in Shahin-Shahr, Isfahan, Iran, to evaluate the EGM2008 and GGMplus Earth geopotential models and a regional geoid model (named IRGeoid10) in terms of agreement to the GPS/leveling observations and introduce the more likely correct model over the case-study area. The results indicate that the EGM2008 model shows a good agreement (2-mm precision on the fitted surface) with the results of the GPS/leveling observations in this very small area. It is notable, however, that the present contribution is mainly of interest from an algorithmic point of view because concrete conclusions cannot be made when comparing the results due to the small case-study area used. By optimal combination of these data sets, the weights of which are estimated by LS-VCE, a geometric surface is presented to approximate the local vertical datum in the case-study region. This surface can convert the ellipsoidal heights, which can be obtained from GPS, to the orthometric heights with high precision. DOI: 10.1061/(ASCE)SU.1943-5428.0000205. © 2016 American Society of Civil Engineers.

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Introduction

In many geodetic applications, height determination using Global Positioning System (GPS) measurements is an alternative to the traditional leveling techniques. In a large country like Iran, spirit leveling is too costly because the establishment of a leveling network covering the entire country becomes time-consuming and expensive. Based on the Gauss-Listing definition, geoid is an equipotential surface of the Earth’s actual gravity field, which best fits the global mean sea level in the least-squares sense (Gauss 1828). At each point, the geoid undulation (also called geoidal height) is the distance between the geoid and ellipsoid measured along the ellipsoidal normal (Hofmann-Wellenhof and Moritz 2006). Determination of a more likely correct local geoid becomes a challenging problem when converting the ellipsoidal heights (obtained from GPS) to orthometric heights (Featherstone et al. 1998). It means that the conversion of GPS heights to orthometric heights needs a precise geoid model to provide geoid heights.

The commonly used procedures that can provide the geoid heights are categorized into three groups: GPS/leveling, gravimetric geoid models, and Earth geopotential models (EGMs). Among them, EGMs provide global geoid heights of the Earth’s gravity field, but geometric and gravimetric models can be developed for various areas, either locally or regionally. Although these methods are considerably different in terms of the physical concepts, reference surfaces, observational and assimilation methods, and their accuracies, they can all be used to transform the ellipsoidal heights into orthometric ones using the following approximate formula:

\[ N \approx h - H \]  \hspace{1cm} (1)

where \( N, h, \) and \( H \) are the geoid undulation, ellipsoidal height, and orthometric height, respectively (Fig. 1). It is possible with sophisticated GPS data processing and precise leveling to obtain a local geometric geoid model using the GPS/leveling observations (Erol and Erol 2013; Featherstone et al. 1998; Zhong 1997). The optimal combination of GPS and leveling observations along with the available geoid models (Featherstone 2000; Soltanpour et al. 2006; You 2006) and the ultimate accuracy of such a combination is still a challenging problem (Fotopoulos 2005; Fotopoulos et al. 2003). This issue is the subject of discussion in the present contribution.

From the last decade of the twentieth century different aspects of fitting a continuous surface to discrete GPS/leveling data have been studied, and different techniques have been used for many case studies. Among them, the reader may at least be referred to the inverse distance weighting (Erol and Čelik 2005; Zhan-ji and Yong-qi 1999), multivariable polynomial regression equations...
(MPREs) (Georgopoulos and Telioni 2015; Zhong 1997), geostatistical kriging (Erol and Celik 2005), least-square collocation (Brown et al. 2011; Featherstone and Sproule 2006; Featherstone et al. 2011; Kotsakis and Sideris 1999; Tscherning et al. 2001), artificial neural networks (ANNs) (Erol and Erol 2013; Kavzoglu and Saka 2005), and the adaptive neuro fuzzy inference system (ANFIS) (Erol and Erol 2013).

Among the researchers who have been dealing with the optimal combination of different sources of information for geoid modeling, the reader may be referred to the studies by Jiang and Duquenne (1996) and Kotsakis and Sideris (1999). Fotopoulos (2005) applied the minimum norm quadratic unbiased estimation (MINQUE) technique to the combined adjustment of GPS, leveling, and geoid data to estimate the unknown variance components of the stochastic models and, hence, to obtain more realistic precision for the unknown parameters.

This contribution proposes an approach to optimally combine the available geoid models using the least-squares variance component estimation (LS-VCE) technique. Use is made of the MPREs as a standard technique to produce a geometric surface; it is a best least-squares fit of which the local leveling and GPS observations can significantly contribute to approximating it. Generally speaking, a combination of different sources of information on geoid approximation requires the following two essential tasks:

- The choice for the best parametric model (functional model) to estimate the fitted geoid model as a geometrical surface without any physical meaning; and
- A realistic stochastic model for combining heterogeneous groups of observations in the least-squares adjustment procedure.

This paper is organized as follows. At first the LS-VCE method is briefly explained. The next section describes the problem and the strategy to combine different data sets. The experiment and the available data sets are then introduced. Next, the results are presented and discussed. Finally, the conclusions are made in the last section.

**LS-VCE**

For some geodetic applications, the covariance matrix of observations can be expressed as an unknown linear combination of known cofactor matrices. This holds true when assessing noise characteristics of GPS time series (Amiri-Simkooei et al. 2007). The estimation of the unknown (co)variance components in a linear model is referred to as variance component estimation (VCE). Different methods can be outlined for VCE in the context of a least-squares adjustment including MINQUE (Junhuan et al. 2011; Rao 1971; Sjöberg 1983), the best invariant unbiased estimation (BIQUE) (Koch 1978, 1999), the restricted maximum likelihood estimation (REML) (Koch 1986), and the LS-VCE (Amiri-Simkooei 2007; Teunissen 1988; Teunissen and Amiri-Simkooei 2008), which have been proven that they differ in their formulation and principle used (Amiri-Simkooei 2007). When the distribution of the observables is Gaussian, LS-VCE provides identical results with other VCE methods, such as BIQUE, MINQUE, and REML.

Among the existing VCE methods, the LS-VCE, which in its original form was developed by Teunissen (1988), is used in the present contribution. For applications of LS-VCE to geodetic data series in general and Global Navigation Satellite System (GNSS) in particular, the reader may be referred to Amiri-Simkooei and Tiberius (2007), Amiri-Simkooei et al. (2007, 2009, 2013), Amiri-Simkooei (2007, 2009, 2013), and Fillmer et al. (2014).

This contribution considers another application of the LS-VCE. It has many attractive features. LS-VCE is a simple method because it is formulated using the well-known least-squares principle. The existing body of knowledge on the least-squares theory is directly applied to LS-VCE. This allows one to have unified least-square frameworks to estimate and test the unknown parameters of the functional and stochastic models. The following aspects of LS-VCE are highlighted: (1) measures of discrepancies can simply be obtained in the stochastic model, (2) the covariance matrix of the estimated (co)variance components can be determined, (3) minimum variance estimators can be provided if the weight matrix is taken as the inverse of the covariance matrix, (4) when there is a priori information on the variance components they can simply be incorporated into the stochastic model as hard/weighted constraints, (5) a nonlinear variance component model can be solved using the available nonlinear optimization methods, (6) available robust estimation methods can be generalized to the variance component model (Khodabandeh et al. 2012), (7) estimability of the variance components can be investigated using the S-transformation theory proposed by Baarda (1973), and (8) the problem of negative variance components can be avoided using the nonnegative LS-VCE (Amiri-Simkooei 2016). For detailed information the reader may be referred to Teunissen and Amiri-Simkooei (2008).

Having a linear model of observation equations available, the covariance matrix of the observables can be written as an unknown linear combination of known cofactor matrices. The coefficient of this linear combination are the unknown (co)variance components to be estimated. Therefore, adopting the Gauss-Markov model, the following equations can be considered:

\[ E\{ y \} = Ax; \quad D\{ y \} = Q = \sum_{k=1}^{p} \sigma_k Q_k \]

consisting of the functional and stochastic model, respectively. In the preceding equations, \( y = m \times 1 \) vector of observables; \( x = n \times 1 \) vector of unknown parameters; \( A \) is the \( m \times n \) design matrix assumed to have full column rank; \( Q = m \times m \) covariance matrix of the observables; and \( E \) and \( D = \) expectation and dispersion operators, respectively. The matrices \( Q_k, k = 1 \ldots p \) are the symmetric and linearly independent cofactor matrices. Parameters \( \sigma_k, k = 1 \ldots p \) are the unknown variance components to be estimated using the LS-VCE. The estimated variance components then read \( \bar{\sigma} = N^{-1}l \), where \( N \) is the \( p \times p \) matrix; and \( l \) is the \( p \times 1 \) vector. The entries of \( N \) and \( l \) are obtained from the following equations, respectively (Amiri-Simkooei 2007):

\[ n_{ij} = \frac{1}{2} tr \left( Q_{ij} P_{ij} Q_{ij}^{-1} P_{ij} Q_{ij}^{-1} \right) \]

and
Fig. 2. Algorithm for implementation of LS-VCE (Note: \( i = \) iteration number; \( d = \) differences between estimated variance components in successive iterations; \( \varepsilon = \) predefined threshold)

\[
\| \lambda_i \| = \frac{1}{2} \varepsilon^T Q_i^{-1} Q_i \varepsilon \tag{4}
\]

where \( i \) and \( j \) run from 1 to \( p \); \( P_i = 1 - A(A^T Q_i^{-1} A)^{-1} A^T Q_i^{-1} \) is an orthogonal projector (Teunissen 2000); and \( \varepsilon = P_i y \) is the \( m \times 1 \) vector of residuals. LS-VCE can directly provide the covariance matrix of the (co)variance estimates as \( Q_\sigma = N^{-1} \). Fig. 2 presents the LS-VCE algorithm in more detail. It starts with choosing a priori variance components for \( i = 0 \) and continues until the differences between the estimated variance components in two successive iterations become less than the predefined threshold \( \varepsilon \). For more information about the LS-VCE theory, the reader may be referred to Teunissen and Amiri-Simkooei (2008) and Amiri-Simkooei (2007).

Problem Description

Assume a network of colocated GPS/leveling points of which the leveling segments (lines between points) and the GPS baselines are measured. On the other hand, a number of geoid models are available in the area of study that provide the geoid undulations at the network points. The goal is to find a parametric model for the geoid undulations based on these heterogeneous categories of observations. Such observations can optimally be combined using the VCE, assuming individual variance components for each of the data sets.

The well-known relationship between GPS, orthometric, and geoid height, (i.e., \( N \cong h - H \)), results in the following linear model of observation equations:

\[
E[y] = Ax \tag{5}
\]

which is decomposed into three classes of equations corresponding to the leveling observations, GPS observations, and those obtained from the available geoid models. They are of the form

\[
\Delta H_{ij} = H_j - H_i \\
\Delta h_{ij} = H_j - H_i + N_{ij}^{\text{fitted}} - N_{ij}^{\text{true}} \\
N_i = N_i^{\text{fitted}}
\]

where \( N_{ij}^{\text{fitted}} \) are geoid undulations, which can be estimated using the following polynomial regression model:

\[
N_i^{\text{fitted}}(\varphi, \lambda) = \sum_{m=0}^{i} \sum_{n=0}^{i} a_{mn}(\varphi - \varphi_0)^m(\lambda - \lambda_0)^n\tag{7}
\]

in which \( \varphi, \lambda = \) coordinates; and \( \varphi_0, \lambda_0 = \) coordinates of the mid-point of the network. The coefficients \( a_{mn} \) are unknown to be estimated.

Therefore, in Eq. (5) \( x \) is the vector of unknown parameters including the orthometric heights and the parameters of the polynomial model \( \{a_{mn}\} \), which are to be estimated in the least-squares adjustment; \( y \) is the vector of observations consisting of the measured height differences \( \Delta H, \Delta h \) (from leveling and GPS campaigns, respectively) and the available geoid heights \( N \) (from the existing geoid models). Prior to the least-squares adjustment process, orthometric corrections should be applied to the differential spirit leveling observations \( \Delta \) to obtain orthometric heights (differences) in Eq. (1). This correction is necessary because spirit leveling measurements depend on the leveling path, and the height differences cannot necessarily become zero in the closed loops (Heiskanen and Moritz 1967; Jekeli 2000).

Assuming that the three different groups of height data are uncorrelated, the covariance matrix of the observations is considered to have the following block diagonal form:

\[
Q_y = \begin{bmatrix}
C_H & 0 & 0 & 0 & 0 \\
0 & C_\delta & 0 & 0 & 0 \\
0 & 0 & C_N & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & C_N
\end{bmatrix}
\tag{8}
\]

The covariance matrix of Eq. (8) can be reformulated to have the stochastic model as

\[
Q_y = \sigma_H^2 Q_H + \sigma_\delta^2 Q_\delta + \sigma_N^2 Q_N + \cdots + \sigma_N^2 Q_N
\tag{9}
\]

where \( Q_H, Q_\delta, \) and \( Q_N \) on the right side are known positive semidefinite cofactor matrices for the observed orthometric height differences, GPS height differences, and the available geoid height data sets, respectively. The coefficients \( \sigma_\delta^2 \) are the unknown variance components to be estimated. The covariance between the geoid models can also be estimated considering their corresponding cofactor matrices, which are insignificant in this investigation.

Eqs. (8) and (9) show that to make the final covariance matrix \( Q_y \), the matrices \( C_i \) are to be known. Because there is usually no prior information on these matrices, the simplest choice is to consider them as identity matrices. Therefore, the cofactor matrices \( Q_i \) will have a block diagonal structure with unit and zero entries on their diagonal. For each group of height data only the diagonal entries corresponding to that data are set to 1, and the rest are filled with zeros. Under this assumption, the estimated coefficients \( \sigma_\delta^2 \) can be interpreted as the variance of the data for that group, but it is possible to make an exception for \( Q_H \) because errors in the leveling data usually propagate with the square root of the distance (Vaníček and Grafarend 1980). Therefore the diagonal elements of \( Q_H, \) those related to the leveling data, are proportional to the distance (in kilometers) between the leveling points.

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Experiment and Available Data Sets: GPS/Leveling Network

A network including 249 benchmarks was designed and established by a team of colleagues in a small (~100 km²) area named Shahin-Shahr located north of Isfahan, Iran, as an urban infrastructure to provide some GPS/leveling benchmarks for future geodetic works ordered by the Shahin-Shahr municipality. It covers the entire region with a relatively uniform spatial distribution, as shown in Fig. 3. GPS and precise leveling observations are measured on these points and the network segments. Such a distribution and colocating of GPS and leveling benchmarks make this network suitable for the investigation. The leveling measurements were collected in the summer of 2013. Leveling was performed with the standard of Iranian first-order leveling network (standard deviation of about 2 mm/km1/2). Precise leveling is done using the Leica DNA03 digital level. The nominal standard deviation of the height measurements using the Leica DNA03 (Leica Geosystems AG, Heerbrugg, Switzerland), reported by the manufacturer, is 2 mm/km in double run. The network is small and very dense, almost a point per 300 m. All numerical results and evaluation, in this paper, will refer to this network.

The leveling loops were designed between the network points, and precise leveling was performed along the 290 segments in forward and backward modes. The leveling network was connected to the precise leveling Iranian network using the BUCD1014 benchmark and precise leveling observations are measured on these points and the network segments. Such a distribution and colocating of GPS and leveling benchmarks make this network suitable for the investigation. The leveling measurements were collected in the summer of 2013. Leveling was performed with the standard of Iranian first-order leveling network (standard deviation of about 2 mm/km1/2). Precise leveling is done using the Leica DNA03 digital level. The nominal standard deviation of the height measurements using the Leica DNA03 (Leica Geosystems AG, Heerbrugg, Switzerland), reported by the manufacturer, is 2 mm/km in double run. The network is small and very dense, almost a point per 300 m. All numerical results and evaluation, in this paper, will refer to this network.

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the gravity field functionals including geoid undulation due to the use of dense satellite observations with uniform coverage. Therefore, one can compute geoid undulations in spherical approximation from a set of these coefficients by (Heiskanen and Moritz 1967)

$$N = \frac{GM}{r^\gamma} \sum_{n=2}^{N_{\text{max}}} \sum_{m=0}^{n} [\mathcal{T}_{nm} \cos m\lambda + \mathcal{S}_{nm} \sin m\lambda] \mathcal{P}_{nm}(\cos \theta)$$

where $\lambda$ is geocentric latitude and longitude of the point in which $N$ is to be determined; $\mathcal{T}_{nm}, \mathcal{S}_{nm}$ = fully normalized spherical geopotential coefficients of the anomalous potential; $\mathcal{P}_{nm}$ = fully normalized associated Legendre functions; and $N_{\text{max}}$ = maximum degree of the geopotential model. Over the last decade, continuous efforts have been made to develop more accurate models in parallel with the progression of satellite gravimetric methods and launching of gravity field mapping missions, such as CHAMP, GRACE, and GOCE. In addition, increased density and improvements in quality of the ground data and new computational resources available for numerical modeling studies have significantly improved the quality of the models of the Earth’s gravity. Some of these efforts are highlighted as follows: Förste et al. (2006, 2008a, b), Lemoine et al. (1998), Rapp (1998), and Tapley et al. (2005, 2007).

Among the developments in geopotential modeling, one of the best outcomes is the release of EGM2008 by the National Geospatial-Intelligence Agency (NGA), a model complete to degree and order 2159, with additional spherical harmonic coefficients up to degree 2190 and order 2159. The accuracy goal for EGM2008 was set to 15 cm by NGA, which has been met (Pavlis et al. 2012). The gravity anomaly grid implemented in the development procedure of EGM2008 was formed by merging terrestrial, altimetry-derived, and airborne gravity data, which were available with relatively high accuracy over Iran (Pavlis et al. 2008, p. 11). The discrepancies between EGM2008 geoid undulations and independent GPS/leveling values were estimated to be on the order of $\pm 5$ to $\pm 10$ cm (Pavlis et al. 2012). In this research EGM2008 is used as one of the most common geopotential models.

Use is also made of one of the recent global gravity models that uses gravity from the GOCE satellite mission. This model is the GGMplus, which is a global gravity model released by Hirt et al. (2013) as a contribution of Curtin University and Technical University Munich, Germany. GGMplus is a composite of GRACE and GOCE satellite gravity (providing the spatial scales of 10,000 km down to $\sim 100$ km), EGM2008 (scales of 100 to $\sim 10$ km), and short-scale topographic gravity effects ($\sim 10$ km to $\sim 250$ m). The model, which was developed using the advanced computing resources, provides maps and data of the Earth’s gravity, with a 200-m resolution, for all land and near-coastal areas between $\pm 60^\circ$ latitude. Therefore, GGMplus provides a local resolution of the Earth’s gravity field, with near global coverage. The GGMplus gravity field model provides gravity field functionals, which refer to the Earth’s surface, e.g., the Molodenskii quasigeoid heights.

Because of the small area considered in this contribution, these quasigeoid heights can be considered as approximations of the geoid heights. These data sets are freely available from the Western Australian Geodesy Group (2013).

### Gravimetric Geoid Model IRGeoid10

For a brief introduction on the background of geoid modeling in Iran, the last gravimetric geoid models released from 1988 are mentioned. They include the models IRGeoid88 (Weber and Zomorrodian 1988), IRGeoid01 (Ardalan et al. 2001), IRGeoid05a (Safari et al. 2005), IRGeoid05b (Nakavandchi and Soltanpour 2005), IRGeoid06 (Kiamehr 2006), and finally IRGeoid10 (Hatam Chavari 2010), which were developed over the last three decades to improve the precision of geoids over Iran. The most recent model, IRGeoid10, is a gravimetric geoid introduced by Hatam Chavari (2010) for Iranian territory ($25^\circ \leq \phi \leq 40^\circ$ and $44^\circ \leq \lambda \leq 64^\circ$) on a $3 \times 3$-arc-min grid using the remove-restore technique and the Helmert’s second condensation method. The data sets of this model consist of gravity anomalies, global geopotential models, Shuttle Radar Topography Mission Digital Terrain Model (SRTM DTM), and GPS/leveling data. Terrestrial gravity anomalies are obtained from the integration of different sources, the data supplied by Bureau Gravimétrique International (BGI), the National Cartographic Center (NCC) of Iran, and satellite altimetry-based gravity anomalies for gap areas. For the long wavelengths the model GOCO01S coupled by EGM2008 up to degree and order 360/360 [GOCO01SEGMO8(360), as Hatam Chavari (2010) named it] is used. Another data set used is a high-resolution digital terrain model, $3 \times 3$-arc-sec, SRTM data (obtained from the Shuttle Radar Topography mission) required to compute the terrain correction and indirect effect on geoid surface while reducing gravity anomalies to the geoid surface. A total number of 819 GPS/leveling points spread over Iran are used for validation of the computed geoid. These points are provided by NCC in three different types since 1988. The accuracy of GPS ellipsoidal heights of these three data sets are estimated to be $\pm 0.25$ m, $\pm 1$ cm, and $\pm 3$ cm, respectively. The standard deviations for the first-order leveling lines and loops over Iran are estimated as $\pm 1.43$ mm/√km and $\pm 3.26$ mm/√km, respectively. The overall accuracy of IRGeoid10 gravimetric geoid, estimated using 819 GPS/leveling points, is about $\pm 26$ cm (Hatam Chavari 2010). The IRGeoid10A is a fitted geometric geoid, adjusted at the 819 GPS/leveling points, using a 4-parameter model suitable for orthometric height determination by GPS (that is a zero height reference rather than a geopotential surface).

### Results and Discussions

#### Data Set Comparisons

This section is started by comparing the absolute GPS/leveling-derived values for the geoid heights of network points with their
corresponding values obtained from the models EGM2008, GGMplus, IRGeoid10, and IRGeoid10A. The GPS/leveling geoid heights in this comparison are obtained from the adjusted ellipsoidal and orthometric heights of the points. The statistics of the individual height data sets, used in this investigation, are given in Table 1. These geoid heights are plotted in Fig. 4 after sorting them in order of GPS/leveling heights and removing their offsets at the first point. A good agreement is observed between the geoid heights obtained from EGM2008 and those obtained from the GPS/leveling. Except for the presence of some offsets that could be caused by the improper tide consideration, the zero-degree term of the geopotential expansion, a similar pattern is also observed for all geoid models.

Comparison between the GPS/leveling results and the geoid heights of the available models are also performed. The statistics of the differences are tabulated (Table 2). Fig. 5 presents the height variation and spatial variations of the residuals \( N_{\text{GPS/L}} - N_{\text{Model}} \) over the case-study region. They are plotted versus the heights, latitudes, and longitudes of the GPS/leveling benchmarks. As already mentioned, clear shifts can be seen for most of the data sets. Also, slight slopes are observed on the residuals, indicating that the geoid slopes are slightly different from that derived from the GPS/leveling observations.

**LS-VCE and Surface Approximation**

The authors now demonstrate the implementation of LS-VCE on different data sets. The results are then described and analyzed in detail.

**Implementation**

As already pointed out previously, based on Eq. (6) a parametric model denoted by \( N_{\text{fitted}} \) is considered to form the model of the observation equations (functional model). Two 2-dimensional (2D) polynomial regression equations are chosen in linear and quadratic forms. The results are then presented for these two forms.

With this assumption, \( N_{\text{fitted}} \) in Eq. (6) can be rewritten as a first-order polynomial

\[
N_{\text{fitted}} = a_0 + a_1 (\varphi_i - \varphi_0) + a_2 (\lambda_i - \lambda_0)
\]

and a second-order polynomial

\[
N_{\text{fitted}} = a_0 + a_1 (\varphi_i - \varphi_0) + a_2 (\lambda_i - \lambda_0) + a_3 (\varphi_i - \varphi_0)^2 + a_4 (\lambda_i - \lambda_0)^2 + a_5 (\varphi_i - \varphi_0) (\lambda_i - \lambda_0)
\]

**Table 2. Statistics of Differences \( N_{\text{GPS/L}} - N_{\text{Model}} \) over the Case-Study Region**

| \(~N_{\text{GPS/L}} - N_{\text{Model}}~| | Max (m) | Min (m) | Mean (m) | Standard deviation (m) |
|---|---|---|---|---|
| \(~N_{\text{GPS/L}} - N_{\text{EGM2008}}~| | 0.002 | -0.083 | -0.039 | 0.013 |
| \(~N_{\text{GPS/L}} - N_{\text{GGMplus}}~| | -0.625 | -0.761 | -0.682 | 0.023 |
| \(~N_{\text{GPS/L}} - N_{\text{IRGeoid10}}~| | 0.534 | 0.428 | 0.481 | 0.020 |
| \(~N_{\text{GPS/L}} - N_{\text{IRGeoid10A}}~| | 0.317 | 0.183 | 0.246 | 0.024 |

**Fig. 4.** Comparison of different data sets over the case-study region
To make a correct link between Eqs. (5) and (6), it seems necessary to present the explicit expression of the design matrix $A$. Briefly, the rows of the design matrix are presented according to the three classes of equations in Eq. (6), assuming $N_{\text{fitted}}$ is a first-order polynomial. They read

$$y \rightarrow A(p) = \begin{bmatrix}
\frac{\partial}{\partial H_1} & \frac{\partial}{\partial H_2} & \cdots & \frac{\partial}{\partial H_i} & \frac{\partial}{\partial H_{i+1}} & \cdots & \frac{\partial}{\partial H_{j+2}} & \cdots & \frac{\partial}{\partial H_j} & \frac{\partial}{\partial H_{j+1}} & \cdots & \frac{\partial}{\partial a_0} & \frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} \\
\Delta H_{ij} & A(p) = [0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 & 0] \\
\Delta h_{ij} & A(p) = [0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & (\varphi_i - \varphi_j) & (\lambda_i - \lambda_j)] \\
N_i & A(p) = [0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & (\varphi_i - \varphi_0) & (\lambda_i - \lambda_0)]
\end{bmatrix}$$

(17)

where the first row of the preceding equation shows the parameter types indicated in the design matrix; and $A(p)$ indicates the $p$th row of the design matrix $A$. Values 1 and $-1$ in the rows correspond to $\Delta H_{ij}$ and $\Delta h_{ij}$ placed at the $i$th and $j$th columns. The three last columns of the design matrix are due to the three unknown parameters of the first-order polynomial.

The stochastic model of the different geoid height data sets (EGM2008, GGMplus, IRGeoid10, and IRGeoid10A) then reads

$$Q_{\gamma} = \sigma_h^2 Q_h + \sigma_e^2 Q_e + \sigma_{N_{\text{MEG2008}}}^2 Q_{N_{\text{MEG2008}}} + \sigma_{N_{\text{GGMplus}}}^2 Q_{N_{\text{GGMplus}}}$$

$$+ \sigma_{N_{\text{IRGeoid10}}}^2 Q_{N_{\text{IRGeoid10}}} + \sigma_{N_{\text{IRGeoid10A}}}^2 Q_{N_{\text{IRGeoid10A}}}$$

(18)

where the unknown coefficients $\sigma_{\gamma}^2$ are the variance components to be estimated. The cofactor matrices are assumed to be block diagonal matrices. Therefore, it is possible to make an evaluation of the precision of these data from the estimated variance components.

The initial values of all variance components are assumed to be equal to 1 for the implementation of the iterative LS-VCE algorithm. The iterations continued until practically identical estimates would be reached for the variance components in successive iterations or below a predefined threshold for their differences. The criterion to choose the threshold is to achieve submillimeter precision for the standard deviations. Therefore, the threshold in Fig. 2 was selected to be $\epsilon = 10^{-6}$.

**Results**

The convergence of the variance components is illustrated in Fig. 6. The precision of the orthometric height differences, ellipsoidal height differences, and EGM2008-, GGMplus-, IRGeoid10-, and IRGeoid10A-derived geoid heights are then estimated for the two (first- and second-order) polynomials (Table 3). Note that the estimated $\sigma_{\gamma}^2$ coefficients in Eq. (18) indicate variances (in square meters), and the results presented in this table are their square roots expressed as standard deviations. The covariance matrix of the...
A few observations from the results presented in Table 3 are highlighted. The estimated precision of the orthometric heights (from leveling) and the ellipsoidal heights (from GPS) is about 2 mm/km$^{1/2}$ and 1 cm, respectively. These results are nearly independent from the functional model used with either the first- or second-order polynomial. When comparing the geoid models one observes that they provide more or less similar results with standard deviations less than 2 cm. The best results are obtained for the EGM2008 model, which provides a standard deviation of 1 cm over the case-study region. It then reduces to 2 mm when using a second-order polynomial. Such a large improvement for EGM2008 indicates that EGM2008, in this area, can best be fitted to a second-order polynomial, and that it is still compatible with GPS/leveling results. This holds also to some extent for IRGeoid10. At first it may appear that the estimated value of 2 mm is optimistic for EGM2008. It is notable that the estimated standard deviations express only the geoid slopes because the biases or shifts (cf., Figs. 4 and 5) of the different geoid models have not been included in the VCE; they are shifted to have identical mean values. The shifts applied fulfill the requirements for the application considered because they can partly compensate the error sources, such as the improper tide consideration, zero-degree term of geopotential expansion, and geoid to quasigeoid separation. The second reason why such small values of standard deviation are obtained (over all models) is due to the small case-study area, which can be best fitted to a second-order polynomial.

In Table 5 the RMS of the differences of geoid models to the final fitted model are presented. A comparison of Tables 3 and 5 indicates a correlation between the estimated variance components and the differences of the geoid models and the fitted model, which further confirms that the estimated variance components can then be computed directly from the inverse of the normal matrix $N$ [Eq. (3)]; the square root of its diagonal entries are tabulated in Table 4 as the precision of the variance components.

<table>
<thead>
<tr>
<th>Polynomial order</th>
<th>$\sigma_h$ (mm)</th>
<th>$\sigma_h^*$ (mm)</th>
<th>$\sigma_N$ (mm)</th>
<th>$\sigma_N^*$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>2.2</td>
<td>10.4</td>
<td>11.6</td>
<td>15.9</td>
</tr>
<tr>
<td>Second</td>
<td>2.2</td>
<td>10.7</td>
<td>1.8</td>
<td>17.9</td>
</tr>
</tbody>
</table>

Note: $\sigma_h$ = orthometric height differences (mm/km$^{1/2}$); $\sigma_h^*$ = ellipsoidal height differences; $\sigma_N$ = EGM2008-derived geoid height; $\sigma_N^*$ = GGMplus-derived geoid height; $\sigma_N$ = IRGeoid10-derived geoid height; and $\sigma_N^*$ = IRGeoid10A-derived geoid height.

<table>
<thead>
<tr>
<th>Polynomial order</th>
<th>$\sigma_N^*$ (mm)</th>
<th>$\sigma_N^*$ (mm)</th>
<th>$\sigma_N^*$ (mm)</th>
<th>$\sigma_N^*$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>0.0003</td>
<td>0.0067</td>
<td>0.0003</td>
<td>0.0028</td>
</tr>
<tr>
<td>Second</td>
<td>0.0004</td>
<td>0.0072</td>
<td>0.0003</td>
<td>0.0079</td>
</tr>
</tbody>
</table>

Note: $\sigma_N^*$ = orthometric height differences (mm/km$^{1/2}$); $\sigma_N^*$ = ellipsoidal height differences; $\sigma_N^*$ = EGM2008-derived geoid height; $\sigma_N^*$ = GGMplus-derived geoid height; $\sigma_N^*$ = IRGeoid10-derived geoid height; and $\sigma_N^*$ = IRGeoid10A-derived geoid height.
The estimated coefficients of the polynomial equations and the orthometric heights as well as the geoid heights of the network points. The estimated coefficients of the polynomials, which approximate geoid as a geometric surface patch, along with their standard deviations are presented in Table 6, which also presents the 95% confidence interval for the estimated parameters. All estimated parameters are statistically significant (except for $a_3$), indicating that a second-order polynomial can best be fitted to the data set considered. The estimated surfaces are illustrated in Fig. 7.

### Summary and Conclusions

The researchers implemented the LS-VCE to combine the geodetic observations of the GPS and orthometric heights as well as the geoid heights obtained from the existing models. The goal of this research is to present a methodology for combining different data sets of available geoid models. The authors also evaluate the precision of the available geoid data sets on the case-study area and combine them in an optimal least-squares sense. Two global geopotential models, two local gravimetric/adjusted geoid models, and observations of a GPS/leveling network were combined using the LS-VCE method. The functional part of the model considered a first-order and a second-order 2D polynomial to express the geometric surface of the local geoid in the study area. Compared with the other models, the EGM2008 model produced the most precise geoid height data, and its results closely follow those obtained by the GPS/leveling network. The estimated standard deviation of the EGM2008 is about 2 mm. In the study area, this issue suggests that this model is suitable for most geodetic applications. The estimated standard deviation of GGMplus is larger than that of EGM2008 (17 mm versus 2 mm). This may make sense because the GGMplus provides quasi-geoid heights. The results also indicated that a second-order polynomial is superior over the first-order polynomial. This is mainly because some of the variance components became significantly smaller when using a second-order polynomial. It is also observed that most of the estimated parameters were statistically significant, indicating that a second-order polynomial could best be fitted to the data set considered. In conclusion, such a polynomial can express the final geometric geoid model, which was obtained from the combination of all available data sets with suitable weighting.

### Acknowledgments

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**Table 6. Estimated Coefficients of Two Polynomials along with Their Precision for Suggested Geoid Models**

<table>
<thead>
<tr>
<th>Polynomial coefficient</th>
<th>First-order polynomial</th>
<th>Standard deviation</th>
<th>95% confidence interval</th>
<th>Second-order polynomial</th>
<th>Standard deviation</th>
<th>95% confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>-0.247640</td>
<td>0.007</td>
<td>[-0.014, 0.014]</td>
<td>0.175970</td>
<td>0.025</td>
<td>[-0.490, 0.490]</td>
</tr>
<tr>
<td>$a_1$</td>
<td>3.810069</td>
<td>0.018</td>
<td>[-0.035, 0.035]</td>
<td>1.462342</td>
<td>0.132</td>
<td>[-0.259, 0.259]</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-2.574862</td>
<td>0.027</td>
<td>[-0.053, 0.053]</td>
<td>-3.889717</td>
<td>0.205</td>
<td>[-0.402, 0.402]</td>
</tr>
<tr>
<td>$a_3$</td>
<td></td>
<td></td>
<td></td>
<td>3.247576</td>
<td>0.175</td>
<td>[-0.343, 0.343]</td>
</tr>
<tr>
<td>$a_4$</td>
<td></td>
<td></td>
<td></td>
<td>-0.047245</td>
<td>0.543</td>
<td>[-1.064, 1.064]</td>
</tr>
<tr>
<td>$a_5$</td>
<td></td>
<td></td>
<td></td>
<td>3.648959</td>
<td>0.526</td>
<td>[-1.031, 1.031]</td>
</tr>
</tbody>
</table>

**Fig. 7.** Estimated geometric geoid surfaces: (a) first-order polynomial; (b) second-order polynomial

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